Codes from Hall planes of even order

Key, J. D.; McDonough, Thomas; Mavron, V.C.

Published in:
Journal of Geometry

DOI:
10.1007/s00022-013-0189-8

Publication date:
2014

Citation for published version (APA):
Key, J. D., McDonough, T., & Mavron, V. C. (2014). Codes from Hall planes of even order. *Journal of Geometry*, *105*(1), 33-41. https://doi.org/10.1007/s00022-013-0189-8

Codes from Hall planes of even order

J. D. Key*, T. P. McDonough†and V. C. Mavron‡
Institute of Mathematics, Physics and Computer Science,
Aberystwyth University, Aberystwyth SY23 3BZ, U.K.

31st October, 2013

Abstract

We show that the binary code C of the projective Hall plane $\mathcal{H}_{q^2}$ of even order q^2 where $q = 2^t$, for $t \geq 2$ has words of weight $2q^2$ in its hull that are not the difference of the incidence vectors of two lines of $\mathcal{H}_{q^2}$; together with an earlier result for the dual Hall planes of even order, this shows that for all $t \geq 2$ the Hall plane and its dual are not tame. We also deduce that $\dim(C) > 3^{2t}+1$, the dimension of the binary code of the desarguesian projective plane of order 2^{2t}, thus supporting the Hamada-Sachar conjecture for this infinite class of planes.

Keywords: Non-desarguesian planes, Hamada-Sachar conjecture, codes
Mathematics Subject Classifications (2010): 94B05, 51A35, 05B05

1 Introduction

The p-ary code of a projective plane Π of order n is the row span over $\mathbb{F}_p$ of an incidence matrix for Π. If n is divisible by p the code has minimum weight $n + 1$ and the minimum words are the scalar multiples of the incidence vectors of the lines; this is an early result in the study of codes from planes, and is easily proved (see [1, Theorem 6.3.1]). For desarguesian planes $\Pi = PG_2(\mathbb{F}_q)$, where the order is $q = p^t$, p a prime, if C is the code of Π over $\mathbb{F}_p$, and $C^\perp$ its dual (orthogonal), then $C \cap C^\perp$, called the hull of Π or C, has minimum weight $2q$ and the minimum words are the scalar multiples of the differences of the incidence vectors of two lines: this is from work of Delsarte, Goethals and MacWilliams, and quoted in full in [1, Chapters 5,6], with the relevant references. This property led to the concept of a tame projective plane which was introduced in [1, Definition 6.9.1] as a tool in the search for a coding-theoretic classification of projective planes. Thus a projective plane of order n is tame at a prime p, where p divides n, if the hull of the plane has minimum weight $2n$ and the vectors of weight $2n$ are precisely the scalar multiples of the differences of the incidence vectors of two lines. It follows that the desarguesian planes are tame, but at present nos other planes have been shown to be tame, and many of small order have been shown not to be tame, either because the minimum weight of the hull is not $2n$ (see [5]) or, more frequently, that there are

*keyj@clemson.edu
†tpd@aber.ac.uk
‡vcm@aber.ac.uk

words of weight $2n$ that are not scalar multiples of the differences of the incidence vectors of two lines. We show here that the Hall planes of even order 2^{2t} for $t \geq 2$ are not tame by exhibiting words of weight 2^{2t+1} in the binary hull that are not differences of the incidence vectors of two lines. Using also a result in [7, Corollary 3], this shows that the even order Hall planes and their dual planes are not tame for all even orders $n > 4$.

Another outstanding conjecture concerning codes from projective planes is the Hamada-Sachar conjecture that the desarguesian planes always have codes of smaller dimension than those of non-desarguesian planes: see [1, Conjecture 6.9.1] (quoted below in Section 2, Conjecture 1), or, more recently, [12]. In practice this has been demonstrated computationally for many individual planes of small order. We show here that the Hall planes of even order $q = 2^t$ have binary codes with dimension greater than that of the desarguesian plane, i.e. greater than 3^t+1, thus reaffirming the conjecture for an infinite class of planes.

We will prove the following proposition through a series of lemmas.

Proposition 1 *Let $q = 2^t$, $t \geq 1$, and $\Pi = PG_2(\mathbb{F}_{q^2})$. Let δ be a Baer segment on a line ℓ. If P is any point in δ and ℓ_1, ℓ_2 any two lines through P other than ℓ, then there is a set of $\frac{q^2}{2}$ points $P_i \in \ell \setminus \delta$ and q^2 lines $m_{i,1}, m_{i,2}$ such that $P_i \in m_{i,1}, m_{i,2}$ for $1 \leq i \leq \frac{q^2}{2}$ and*

$$v^{\ell_1} + v^{\ell_2} = s \sum_{i=1}^{\frac{q^2}{2}} (v^{m_{i,1}} + v^{m_{i,2}}).$$

If $\mathcal{L}$ denotes the set of lines of Π that meet ℓ in $\ell \setminus \delta$, and $E = \langle v^m \mid m \in \mathcal{L} \rangle$ over $\mathbb{F}_2$, then $\dim(E) = 3^{2t} - 2^t$.

If $\mathcal{H}$ is the Hall plane of order q^2 then $\mathrm{Hull}(C_2(\mathcal{H}))$ contains words of weight $2q^2$ that have support the symmetric difference of two Baer subplanes that intersect in a line, and for $t \geq 2$ neither $\mathcal{H}$ nor its dual $\mathcal{H}'$ is tame. Further, $\dim(C_2(\mathcal{H})) > \dim(C_2(\Pi)) = 3^{2t} + 1$.

The proof is in Section 3. Section 2 gives some definitions and background results and questions. The last section mentions some computational observations we have made for planes of low order.

2 Terminology and notation

Standard terminology for designs and codes is used as in [1] and for planes as in [6]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. A 2-$(n^2+n+1, n+1, 1)$ design, for $n \geq 2$, is a **finite projective plane of order** n. We write $PG_{2,1}(\mathbb{F}_q)$ or $PG_2(\mathbb{F}_q)$ for the desarguesian projective plane, i.e. the design of points and 1-dimensional subspaces of the projective space $PG_2(\mathbb{F}_q)$. Similarly, $AG_{m,t}(\mathbb{F}_q)$ will denote the design of points of the affine space $AG_m(\mathbb{F}_q)$ and t-flats, where $t \geq 1$.

If Π is a projective plane of square order n^2, a subplane π of Π of order n is called a **Baer subplane**. Lines of Π meet π in 1 or $(n+1)$ points. If a line of Π meets π in a set δ of $n+1$ points, δ is called a **Baer segment**. An **oval** or **hyperoval** in a projective plane of even order n is a set of $n+2$ points such that lines meet the set in 0 or two points.

The **code C_F of the design** $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. We take F to be a prime field $\mathbb{F}_p$ and the prime must divide the order of the design, i.e. n for a finite plane of order n. Denoting the incidence vector of a subset $\mathcal{Q}$ of points by $v^{\mathcal{Q}}$, we have $C_F = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F.

A linear code over $\mathbb{F}_q$ of length n, dimension k, and minimum weight d, is denoted by $[n, k, d]_q$. If c is a codeword then the **support** of c is the set of non-zero coordinate positions of c, and the **weight** of a vector is the size of its support. For any code C, the **dual** or **orthogonal** code $C^\perp$ is the orthogonal under the standard inner product, i.e. $C^\perp = \{v \in F^n \mid (v, c) = 0 \text{ for all } c \in C\}$. The **hull** of a design with code C is $C \cap C^\perp$, written $\mathrm{Hull}(\mathcal{D})$ or $\mathrm{Hull}(C)$.

The following is in [1, Theorem 6.3.1]:

Result 1 *Let Π be a projective plane of order n and let p be a prime dividing n. Then the minimum-weight vectors of $C_p(\Pi)$ are precisely the scalar multiples of the incidence vectors of the lines. Further,* $\mathrm{Hull}_p(\Pi) = \langle v^L - v^M \mid L \text{ and } M \text{ lines of } \Pi \rangle$.

The situation for the codes from desarguesian planes is quoted in [1, Theorem 6.4.2]:

Result 2 *Let p be any prime, $q = p^t$, and $\Pi = PG_2(\mathbf{F}_q)$. Then $C_p(\Pi)$ is a $[q^2+q+1, \binom{p+1}{2}^t + 1, q+1]_p$ code. The minimum-weight vectors of $C_p(\Pi)$ and of $C_p(\Pi) + C_p(\Pi)^\perp$ are the scalar multiples of the incidence vectors of the lines. The minimum weight of* $\mathrm{Hull}_p(\Pi)$ *is $2q$ and its minimum-weight vectors are the scalar multiples of the differences of the incidence vectors of distinct lines of Π.*

The notion of a **tame plane** was introduced in [1, Section 6.9]:

Definition 1 *A projective plane Π of order n is said to be* **tame** *(or tame at p, where p is a prime dividing n) if* $\mathrm{Hull}_p(\Pi)$ *has minimum weight $2n$ and the minimum-weight vectors of* $\mathrm{Hull}_p(\Pi)$ *are precisely the scalar multiples of the differences of the incidence vectors of distinct lines of Π.*

The **Hamada-Sachar conjecture** [1, Conjecture 6.9.1] is as follows:

Conjecture 1 *Every projective plane of order p^s, p a prime, has p-rank at least $\binom{p+1}{2}^s + 1$ with equality if, and only if, it is desarguesian.*

From [1, Proposition 6.3.3],

Result 3 *If X is the set of points of a Baer subplane of $\Pi = PG_2(\mathbb{F}_{q^2})$, then $v^X \notin C(\Pi)$.*

Using a result of Menichetti [9] the following was obtained in [7, Corollary 3]:

Result 4 *Every dual Hall plane of even square order $n > 4$ has a vector of weight $2n$ in its binary hull that is not the difference of the incidence vectors of two lines.*

3 Proof of Proposition 1

We first give the definition of a Hall plane using the process of derivation by Baer segments: see [1, Section 6.10]. The Hall planes are translation planes and the definition can be found in standard texts, for example [4, 11, 6, 8], or [1, Section 6.10]. Let Π be the desarguesian projective plane $PG_2(q^2)$ of order q^2, where $q = p^e$, p prime. Let δ be a Baer segment of $q+1$ points of a line ℓ_∞ of Π and use this as a **derivation set** to construct the Hall plane $\mathcal{H}_{q^2}$ (or simply $\mathcal{H}$ when the order is clear) of order q^2. Let $\mathcal{B}$ be the set of Baer subplanes of Π that meet ℓ_∞ in δ. Then $|\mathcal{B}| = q^2(q+1)$, and any two of these Baer subplanes meet in one further point, or do not intersect off δ. Any one of these subplanes, together with all those that are mutually disjoint from it off δ, form a set of q^2 subplanes that will form a parallel class of lines in the new plane. There are $q+1$ of these parallel classes of subplanes.

Denote the set of lines of Π that meet ℓ_∞ in $\ell_\infty \setminus \delta$ by $\mathcal{L}_c$. Then $|\mathcal{L}_c| = q^3(q-1)$. Then for any line $m \in \mathcal{L}_c$, $m \setminus \ell_\infty$ will be an affine line of $\mathcal{H}$. Denote this set of affine lines for both Π and $\mathcal{H}$, by $\mathcal{A}_c$.

The other $q^2(q+1)$ affine lines of $\mathcal{H}$ we write as $\mathcal{A}_n$ and these consist of the points of the Baer subplanes in $\mathcal{B}$ with the points in δ removed. To complete the affine Hall plane to the projective Hall plane $\mathcal{H}$, a line at infinity is adjoined using the points on ℓ_∞ as before for the lines in $\mathcal{A}_c$ (so that the lines $\mathcal{L}_c$ are common to both projective planes), and for the remaining points, a point for each class of lines in $\mathcal{A}_n$ that are disjoint, i.e. that as Baer subplanes in Π meet only in the segment δ. We will denote these lines in the projective Hall plane $\mathcal{H}$ by $\mathcal{L}_n$.

The lines of Π that meet ℓ_∞ in δ are denoted by $\mathcal{L}_o$, and, on removing the point in δ, they become affine Baer subplanes of $\mathcal{H}$, of size q^2; if they meet in a point on δ, these planes are disjoint. So in $\mathcal{H}$ they are Baer subplanes that share points only on the line at infinity.

Let $q = 2^t$, $K = \mathbb{F}_{q^2}$, $F = \mathbb{F}_q$, $K^\times = \langle w \rangle$, $F^\times = \langle u \rangle$ where $u = w^{q+1}$. Take $\Pi = PG_2(K)$ and $\mathcal{R}$ a regular oval with nucleus in Π consisting of the conic with equation $X^2 = YZ$ and nucleus $(1,0,0)$. So

$$\mathcal{R} = \{(1, y, y^{-1}) \mid y \in K^\times\} \cup \{(1,0,0), (0,1,0), (0,0,1)\}. \tag{1}$$

Since $\delta = \{(1,0,t) \mid t \in F\} \cup \{(0,0,1)\}$ is a Baer segment, so also, for any $z \in K$, is

$$\delta^* = \{(1,1,t+z) \mid t \in F\} \cup \{(0,0,1)\}$$

since it can be obtained from δ from the collineation given by the matrix $\begin{bmatrix} 1 & 1 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The points $(1,0,0)$ and $(0,1,0)$ are on $\mathcal{R}$ and the line ℓ through them has homogeneous coordinates $(0,0,1)'$. The point $P = (1,1,0)$ is on ℓ and not on $\mathcal{R}$. Clearly $P \in (1,1,t+z)'$ for each $(1,1,t+z) \in \delta^*$. Thus the whole derivation set δ^* is on the line $P' = (1,1,0)'$. We claim that all the other $\frac{q^2}{2}$ secants through P to $\mathcal{R}$ correspond to points not in δ^*.

A line through P, other than ℓ, has the form $(1,1,z)'$ for $z \in K$. Suppose such a line is a secant to $\mathcal{R}$, meeting it at $(1, y, y^{-1})$, where $y \neq 0$, so that $1 + y + zy^{-1} = 0$, or $z = y^2 + y$.

Lemma 1 *For $q = 2^t$, $K = \mathbb{F}_{q^2}$, $F = \mathbb{F}_q$, the set $H = \{y^2 + y \mid y \in K\}$ is an additive subgroup of K of order $\frac{q^2}{2}$, and $H > F$. Further, if $z \notin H$ then the lines $\{(1,1,t+z)' \mid t \in F\}$ are exterior to the regular oval $\mathcal{R}$.*

Proof: Define the map $\varphi : K \to H$ by $\varphi(y) = y^2 + y$. Then φ is an additive homomorphism and $\ker(\varphi) = \{0, 1\}$. Furthermore, if $y \neq z$ then $\varphi(y) = \varphi(z)$ if and only if $y = z + 1$. Clearly $|H| = \frac{q^2}{2}$.

To show that $F < H$, note that every quadratic $X^2 + X + a = 0$, for $a \in F$, must have a solution in K, since K is the unique quadratic extension of F.

Now choose any $z \in K \setminus H$. We claim that for $t \in F$, $z + t \in K \setminus H$. Suppose $z + t \in H$. Then $z + t = y^2 + y$, so $z = y^2 + y + s^2 + s = (y + s)^2 + (y + s) \in H$, where $t = s^2 + s$ by the preceeding paragraphs. Thus none of the lines in $\{(1, 1, t + z)' \mid t \in F\}$ meet $\mathcal{R}$. ■

The notation given above and that in Lemma 1 will be used in all the following.

Lemma 2 *If $z \in K \setminus H$, the line $(1, 1, 0)'$ in $PG_2(\mathbb{F}_{q^2})$, where $q = 2^t$, $t \geq 1$, contains the derivation set δ^*. Let $\ell = (0, 0, 1)'$ be the secant to $\mathcal{R}$ through $(1, 0, 0)$ and $(0, 1, 0)$, and let $\ell_i = P_iQ_i$ for $1 \leq i \leq \frac{q^2}{2}$ be the other secants to $\mathcal{R}$ through $P = (1, 1, 0)$. Then the points ℓ_i' are on $(1, 1, 0)'$ and are not in the set δ^*, and*

$$v^{(1,0,0)'} + v^{(0,1,0)'} = \sum_{i=1}^{\frac{q^2}{2}} (v^{P_i'} + v^{Q_i'}).$$

If $\mathcal{H}$ is a Hall plane formed by taking the derivation set δ^* on the line $(1, 1, 0)'$, then $w = v^{(1,0,0)'} + v^{(0,1,0)'}$ is a word in $C_2(\mathcal{H})$ of weight $2q^2$ which is not the difference of two lines of $\mathcal{H}$ if $t > 1$.

For $t \geq 2$, $q = 2^t$, the Hall planes of order q^2 are not tame, and neither are their duals.

Proof: The $q^2 + 2$ lines in the sum correspond to the points on $\mathcal{R}$. Since it is a hyperoval, every line meets it twice or not at all, which in the dual plane becomes that every point of the dual plane is on two lines or no lines of the $q^2 + 2$ lines in the sum. Thus the sum is zero.

The statement about the Hall plane follows since the lines of Π meeting $(1, 1, 0)'$ in the derivation set become Baer subplanes in $\mathcal{H}$ while all the lines that meet $(1, 1, 0)'$ outside the derivation set are common lines to $\mathcal{H}$ and Π. When two lines of Π are taken that meet in a point of the derivation set, the corresponding Baer subplanes of $\mathcal{H}$ meet in a line, at infinity. Since $(1, 0, 0)'$ and $(0, 1, 0)'$ meet at $(0, 0, 1) \in \delta^*$, we have $w = v^{(1,0,0)'} + v^{(0,1,0)'}$ as a word in $C_2(\mathcal{H})$ of weight $2q^2$ which is not the difference of two lines of $\mathcal{H}$ if $t > 1$.

It follows that for $t \geq 2$ $\mathcal{H}$ is not tame. The statement about the dual Hall planes follows from [7], stated in our Result 4. ■

Note: For $t = 1$, clearly $\mathcal{H} \cong \Pi$ and the support of $w = v^{(1,0,0)'} + v^{(0,1,0)'}$ is in fact the difference of two lines again.

Lemma 3 *For every point $R \in \delta^*$ and every pair of lines $\ell, m \ni R$, where $\ell, m \neq (1, 1, 0)'$, there is a set of $\frac{q^2}{2}$ points R_i for $1 \leq i \leq \frac{q^2}{2}$, on $(1, 1, 0)'$ and a set of pairs of lines $m_{i,1}, m_{i,2}$ such that $R_i \in m_{i,1}, m_{i,2}$ for $1 \leq i \leq \frac{q^2}{2}$, and*

$$v^{\ell} + v^{m} = \sum_{i=1}^{\frac{q^2}{2}} (v^{m_{i,1}} + v^{m_{i,2}}). \tag{2}$$

Proof: First let $R = (0,0,1)$ and $\ell = (1,0,0)'$ and show that any other line (other than $(1,1,0)'$) through R can be taken so that Equation 2 above holds for some choice of points and lines. For this we just need the homologies with axis $(1,1,0)'$ and centre $(0,1,0)$, so that the line $(1,0,0)' \ni R, (0,1,0)$, so it is fixed. Then the homologies fix every point of δ^* and the other lines through R are permuted transitively. Thus ℓ with any other line through R will occur in a sum of the type given in the equation.

Now if $X \neq R$ is another point of δ^* then the elations with centre X and axis $(1,1,0)'$ will permute the lines through R (other than $(1,1,0)'$) transitively, so it follows that any pair of lines through R will occur in a sum of this type.

Finally, to show that any point of δ^* can be taken instead of R, we note that the stabilizer of δ^* in $\mathrm{Aut}(\Pi)$ is certainly transitive on δ^* so R can be mapped to any point while all the other points of δ^* remain in δ^*, so the sum will work for any point of δ^* as asserted. ■

Recall that $\mathcal{L}_c$ is the set of lines that are common to the projective planes Π and $\mathcal{H}$ and that they meet $(1,1,0)'$ (in our construction here) in points of $(1,1,0)' \setminus \delta^*$.

Lemma 4 *Let* $E = \langle v^\ell \mid \ell \in \mathcal{L}_c \rangle$, $q = 2^t$, $t \geq 1$. *Then* $E < C_2(\Pi) \cap C_2(\mathcal{H})$, $\dim(E) = 3^{2t} - 2^t$ *and* $\dim(C_2(\mathcal{H})) > \dim(\Pi)$.

Proof: It is clear that E is in the codes of both planes, as the lines in $\mathcal{L}_c$ are common to the two codes. Now consider what incidence vectors of lines need to be added to E to get the full code $C_2(\Pi)$ of the desarguesian plane. Excluding $v^{(1,1,0)'}$ for now, only one line through any point of δ^* needs to be added to get the whole of $C_2(\Pi)$, since once one line is taken then all the other lines (excluding $v^{(1,1,0)'}$) through the point will be included due to Corollary 3. It is easy to see that at least one line through each point must be taken to get the full code, although $v^{(1,1,0)'}$ could be taken instead of one of these lines. It is clear that the full code is obtained in this way. Thus $\dim(E) = \dim(C_2(\Pi)) - (q+1) = 3^{2t} - 2^t$ from, for example, [1, Theorem 6.4.2].

Now $C_2(\mathcal{H}) \geq E$ and must contain the incidence vector of at least one new line for every one of the $q+1$ new parallel classes, for the same reasons as given above for $C_2(\Pi)$. Suppose we add the vector v^π to E, where π is a line of $\mathcal{H}$ and corresponds to a new point Q on the line at infinity for $\mathcal{H}$. If π^* is another line of $\mathcal{H}$ through Q then we claim that $v^{\pi^*} \notin \langle E, v^\pi \rangle$, recalling that, without Q, both π and π^* are the points of affine Baer subplanes of Π. If $v^{\pi^*} \in \langle E, v^\pi \rangle$ then $v^{\pi^*} = v^\pi + v$, where $v \in E$. Thus $v^{\pi^*} - v^\pi \in E$, which contradicts Result 2, since clearly $v^{\pi^*} - v^\pi \in C_2(\Pi)^\perp$, so this would imply it was in $\mathrm{Hull}(C_2(\Pi))$. Thus for at least one of the $q+1$ points on the line at infinity we need at least two vectors, so $\dim(C_2(\mathcal{H})) \geq \dim(E) + q + 2 > \dim(\Pi)$. ■

The above lemmas sum up to give a proof of Proposition 1.

4 Computational observations

Computational results using Magma [2, 3] with small values of q^2 for q odd indicate that the Hall planes are not tame for the same reason as we have been able to prove for q even. One gets, with the same notation as in Proposition 1,

$$v^{\ell_1} - v^{\ell_2} \in E$$

where E is now generated over the field $\mathbb{F}_p$ where $p \mid q$, and this shows that the plane is not tame since E is a subcode of the p-ary code of the Hall plane $\mathcal{H}$, but the ℓ_1, ℓ_2 are affine Baer subplanes of $\mathcal{H}$. We have not seen a way to prove this as we did in the binary case. In fact we made the following computational observation, but have not proved it apart from what we have shown in this paper:

Observation 1 *Let $\Pi = PG_2(\mathbb{F}_q)$ where $q = p^t$, and p is a prime. Let $C = C_p(\Pi)$, of dimension $\binom{p+1}{2}^t + 1$. Let ℓ be any line of Π, S_1 any set of $p^{t-1} + 1$ points on ℓ and S_2 the remaining $p^t - p^{t-1}$ points of ℓ. Let $\mathcal{L}$ be the set of lines of Π that meet ℓ in S_2, and $E = \langle v^m \mid m \in \mathcal{L} \rangle$. Then*

1. $\dim(E) = \dim(C) - (p^{t-1} + 1)$;
2. *if $P \in S_1$, and m_1, m_2 are two lines (other than ℓ) through P, then $v^{m_1} - v^{m_2} \in E$;*
3. *if $|S_1| > p^{t-1} + 1$ then the previous result need not hold.*

This has been tested with Magma for

- $q = p$ a prime (up to $p = 23$, but one can compute much higher) so that $|S_1| = 2$, in which case the assertion will likely just follow from the Moorhouse basis [10];
- $q = 2^t$ for $1 \leq t \leq 5$, in which case $|S_1| = q/2 + 1$;
- for odd q where $q = 3^2, 5^2, 7^2$, and $q = 3^3$.

Many other planes of odd and even order were examined computationally and none shown to be tame, and most shown not to be tame. In the case of square order q^2, in most cases the words of weight $2q^2$ in the hull not from lines were of the form $v^{\pi_1} - v^{\pi_2}$ where the π_i are Baer subplanes meeting in a line.

All the planes of order 9, and all the known planes of order 16 were examined in [5] and shown not to be tame. In most of the non-translation planes of order 16, words of weight 24 in the hull were found. Most of the known planes of order 25 were shown to be non-tame, but for a few, including the Hughes plane, we were unable to say whether or not they were tame. We also looked at the translation planes of order $q = 27$ for words of the form $v^{S_1} - v^{S_2}$, where S_1 and S_2 were sets of size 27, mostly of the form of affine subspaces $AG_{3,1}(\mathbb{F}_3)$; we found these to exist in more than half the planes, but our search was not exhaustive.

References

[1] E. F. Assmus, Jr and J. D. Key, *Designs and their codes*, Cambridge: Cambridge University Press, 1992, Cambridge Tracts in Mathematics, Vol. 103 (Second printing with corrections, 1993).

[2] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system I: The user language*, J. Symbolic Comput. **24, 3/4** (1997), 235–265.

[3] J. Cannon, A. Steel, and G. White, *Linear codes over finite fields*, Handbook of Magma Functions (J. Cannon and W. Bosma, eds.), Computational Algebra Group, Department of Mathematics, University of Sydney, 2006, V2.13, http://magma.maths.usyd.edu.au/magma, pp. 3951–4023.

[4] P. Dembowski, *Finite geometries*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 44, Berlin, Heidelberg, New York: Springer-Verlag, 1968.

[5] Dina Ghinelli, Marialuisa J. de Resmini, and Jennifer D. Key, *Minimum words of codes from affine planes*, J. Geom. **91** (2008), 43–51.

[6] Daniel R. Hughes and Fred C. Piper, *Projective planes*, Graduate Texts in Mathematics 6, New York: Springer-Verlag, 1973.

[7] J. D. Key and M. J. de Resmini, *Small sets of even type and codewords*, J. Geom. **61** (1998), 83–104.

[8] H. Lüneburg, *Translation planes*, New York: Springer-Verlag, 1980.

[9] Giampaolo Menichetti, *q-archi completi nei piani di Hall di ordine* $q = 2^k$, Lincei - Rend. Sc. fis. mat. e nat. **56** (1974), 518–525.

[10] G. Eric Moorhouse, *Bruck nets, codes, and characters of loops*, Des. Codes Cryptogr. **1** (1991), 7–29.

[11] T. G. Ostrom, *Finite translation planes*, Lecture Notes in Mathematics No. 158, Springer-Verlag, 1970.

[12] V. D. Tonchev, *Finite geometry, designs, codes, and Hamada's conjecture*, Information Security, Coding Theory and Related Combinatorics (Dean Crnković and Vladimir Tonchev, eds.), IOS Press, Amsterdam, 2011, NATO Science for Peace and Security Series, D: Information and Communication Security Vol. 29, pp. 437–448.